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q-Analogues of regularisation theorems for linear and projective representations of the symmetric group

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Abstract

Brundan and Kleshchev have recently proved an analogue of James's 'regularisation theorem' for modular representations of symmetric groups, in the context of projective representations. We prove 'q-analogues' of both theorems, i.e. corresponding theorems concerning canonical basis coefficients for basic representations of quantum affine algebras of types $A^{(1)}_{e-1}$ and $A^{(2)}_{2n}$.

Keywords: Linear and projective representations; Symmetric groups; Canonical bases

1. Introduction

The concept of a *canonical basis* or *global crystal basis* for an integrable highest weight representation of the quantum enveloping algebra $U_q(\mathfrak{g})$ of a symmetrisable Kac–Moody Lie algebra $\mathfrak{g}$ was introduced independently by Lusztig and Kashiwara. In the case where $\mathfrak{g} = \widehat{\mathfrak{sl}}_e$, a fast algorithm for computing the canonical basis of the basic representation was given by Lascoux, Leclerc and Thibon [11], using the Fock space realisation described by Misra and Miwa [14].

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This crystal basis has a variety of applications, one of which is that under the specialisation $q = 1$ the coefficients of this basis give the decomposition numbers for Iwahori–Hecke algebras of type A at a primitive eth root of unity in $\mathbb{C}$; this was proved by Ariki [1]. James's Conjecture asserts that in certain cases, the same is true for Iwahori–Hecke algebras over fields of prime characteristic. The coefficients of the canonical basis prior to specialisation at $q = 1$ (which are polynomials in q with non-negative integer coefficients) are of interest in their own right; they conjecturally describe the Jantzen filtrations of Specht modules for Iwahori–Hecke algebras. In a series of papers [5–7], Chuang, Miyachi and Tan have transferred several results concerning decomposition numbers for Iwahori–Hecke algebras and/or symmetric groups to the canonical basis setting, where they prove 'q-analogues' of these results. In this paper, we prove another such q-analogue, namely of James's regularisation theorem, which (in the symmetric group context) was one of the first general theorems concerning decomposition numbers. It is *de rigueur* when proving such theorems to use only the combinatorics of the Fock space, so that the q-analogue, when combined with Ariki's theorem, affords a new proof of the classical result. We do the same here, although really our proof follows the same line as in [10].

Another regularisation theorem for decomposition numbers has recently been proved by Brundan and Kleshchev, this time in the context of projective representations of symmetric groups in odd characteristic. The combinatorics of such representations are also governed by partitions, and bear a close resemblance to the combinatorics of a Fock space representation of the quantum algebra of type $A^{(2)}_{2n}$. Leclerc and Thibon [12] have given an algorithm for computing the canonical basis for the basic representation of this quantum algebra, and have conjectured that the specialisation at $q = 1$ gives the decomposition numbers for spin representations of symmetric groups in certain cases. Our second main theorem is a regularisation theorem for this canonical basis, and via the conjecture of Leclerc and Thibon it may be viewed as a q-analogue of the theorem of Brundan and Kleshchev.

For the rest of this introduction, we recall the small amount of background which is essential to both of the remaining sections. In Section 2, we prove our q-analogue of James's theorem, and in Section 3 we address the theorem of Brundan and Kleshchev. These two sections are quite separate (and in fact some of the notation and combinatorial notions from Section 2 are re-defined in Section 3), but the structure of the proof in each section is the same.

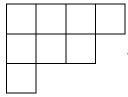
1.1. Partitions and Young diagrams

Here we recall some very basic combinatorial notions which will be useful in the rest of the paper. More complex combinatorics specific to Sections 2 and 3 will be described in those sections.

As usual, a *partition* of r is defined to be a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of non-negative integers whose sum is r. We write the unique partition of 0 as $\varnothing$. We frequently identify a partition λ with its *Young diagram*; this is the set

$$\big\{(j,k) \in \mathbb{N} \times \mathbb{N} \;\big|\; k \leqslant \lambda_j\big\},$$

whose elements we call *nodes*. The Young diagram is conventionally drawn as an array of contiguous boxes (one for each node), with j increasing down the page and k increasing from left to right. For example, the Young diagram of the partition (4, 3, 1) is

 .

A node $\mathfrak{x}$ of λ is called *removable* if $\lambda \setminus \mathfrak{x}$ is the Young diagram of a partition, and a pair $(i, j) \in (\mathbb{N} \times \mathbb{N}) \setminus \lambda$ is called an *addable node of* λ if $\lambda \cup \{\mathfrak{x}\}$ is the Young diagram of a partition. We emphasise that an addable node of λ is not a node of λ.

The *conjugate partition* to λ is the partition λ' (whose Young diagram is) obtained by reflecting the Young diagram of λ in the main diagonal; that is, $\lambda'_j = \max\{k \mid \lambda_k \geqslant j\}$.

Given two partitions λ, μ of r we say that λ *dominates* μ (and write $\lambda \trianglerighteq \mu$) if for every j we have

$$\lambda_1 + \cdots + \lambda_j \geqslant \mu_1 + \cdots + \mu_j.$$

1.2. *Quantum integers and quantum factorials*

Here we recall the notions of quantum integer and quantum factorial, which are important for defining divided powers. If q is an invertible element of any ring, then define the 'quantum integer' $[m]_q = (q^m - q^{-m})/(q - q^{-1})$ and the 'quantum factorial' $[m]_q! = [m]_q[m-1]_q \cdots [1]_q$. In the literature the subscript q is often dropped, but we retain it, since in Section 3 it will take different values.

2. The Fock space of type $A^{(1)}_{e-1}$

For this section we fix an integer e greater than or equal to 2, and examine the quantum algebra of type $A^{(1)}_{e-1}$. For this algebra, the canonical basis of the basic representation describes the decomposition numbers for Iwahori–Hecke algebras of type A at a primitive eth root of unity in $\mathbb{C}$, and we prove an analogue for these canonical bases of James's regularisation theorem.

2.1. *Residues, ladders, and e-restriction*

If λ is a partition, we say that λ is *e-restricted* if $\lambda_j - \lambda_{j+1} < e$ for all j. We define the *residue* of any node $\mathfrak{x} = (j, k)$ of λ to be the integer i such that $0 \leqslant i < e$ and $i \equiv k - j \pmod{e}$. Given any $l \geqslant 1$, we define the lth *ladder* in $\mathbb{N} \times \mathbb{N}$ to be the set

$$\bigl\{(j, k) \in \mathbb{N} \times \mathbb{N} \bigm| (e-1)j + k + 1 - e = l\bigr\}.$$

Note that all the nodes in any ladder have the same residue. Given a partition λ, we define the lth ladder of λ to be the set of nodes of λ lying the lth ladder of $\mathbb{N} \times \mathbb{N}$. If $\lambda \neq \varnothing$ and l is maximal such that the lth ladder of λ is non-empty, we say that the lth ladder is the *outermost* ladder of λ. This ladder consists of removable nodes of λ; we denote by λ^- the partition obtained by removing all these nodes.

It is easy to see that a partition λ is e-restricted if and only if the nodes in each ladder of λ are as far to the left as possible. For any partition λ, we define its *e-restriction* by moving all the nodes in each ladder as far to the left in that ladder as they will go; that is, if there are t nodes in the lth ladder of λ, then we replace these with the leftmost t nodes of the lth ladder in $\mathbb{N} \times \mathbb{N}$. It is a relatively straightforward exercise to see that we obtain the Young diagram of an e-restricted partition in this way, and we denote this partition λ^{R}.

Example. Suppose $e = 5$, and $\lambda = (10, 10, 2)$. Then $\lambda^{\mathrm{R}} = (10, 6, 4, 2)$, as we can see from the following Young diagrams, in which we label each node according to the ladder in which it lies:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&7&8&9&10\\\hline 5&6&7&8&9&10&11&12&13&14\\\hline 9&10\\\cline{1-2}\end{array}\ , \qquad \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&7&8&9&10\\\hline 5&6&7&8&9&10\\\cline{1-6} 9&10&11&12\\\cline{1-4} 13&14\\\cline{1-2}\end{array}\ .$$

2.2. *Representations of symmetric groups and Iwahori–Hecke algebras*

Let $\mathbb{F}$ be a field, and v a non-zero element of $\mathbb{F}$. Suppose that e is the least integer such that $1 + v + \cdots + v^{e-1} = 0$ in $\mathbb{F}$. For any $r \geqslant 0$ let $\mathfrak{S}_r$ denote the symmetric group on r letters, and let $\mathcal{H}_r = \mathcal{H}_{\mathbb{F},v}(\mathfrak{S}_r)$ denote the Iwahori–Hecke algebra of $\mathfrak{S}_r$ over $\mathbb{F}$ with parameter v. In the special case where $v = 1$ in $\mathbb{F}$, $\mathcal{H}_r$ is simply the group algebra $\mathbb{F}\mathfrak{S}_r$ (and e is the characteristic of $\mathbb{F}$). An indispensable reference for the representation theory of $\mathcal{H}_r$ is Mathas's book [13].

For each partition λ of r one defines a *Specht module* S^λ for $\mathcal{H}_r$. If λ is e-restricted, then S^λ has an irreducible socle D_λ, and the modules D_λ give a complete set of irreducible $\mathcal{H}_r$-modules as λ ranges over the set of e-restricted partitions of r. The *decomposition number problem* asks for the composition multiplicities $[S^\lambda : D_\mu]$, where λ and μ are partitions of r with μ e-restricted. The following theorem was first proved by James for the symmetric groups (but with a proof that works for Iwahori–Hecke algebras generally), and was one of the first general results regarding decomposition numbers.

Theorem 2.1. *(See [9, Theorem A].) Suppose λ and μ are partitions of r, with μ e-restricted. Then $[S^\lambda : D_\mu] = 0$ unless $\mu \trianglelefteqslant \lambda^{\mathrm{R}}$, while $[S^\lambda : D_{\lambda^{\mathrm{R}}}] = 1$.*

Remark. Note that our combinatorial conventions are not the most usual ones. The lth ladder is usually defined to be the set

$$\bigl\{(j,k) \in \mathbb{N} \times \mathbb{N} \bigm| j + (e-1)k + 1 - e = l\bigr\},$$

and λ^{R} is the *e-regular* partition obtained by moving nodes up their ladders as far as they will go. Theorem 2.1 is then stated using the simple modules D^μ indexed by e-regular partitions, replacing $\trianglelefteqslant$ with $\trianglerighteqslant$. We have chosen this convention for consistency with the combinatorics of Section 3, and with [13] (though our Specht modules are the duals of those defined by Mathas). Our version of Theorem 2.1 also appears in [8], where 'ladders' are instead called 'ramps.'

2.3. *The quantum algebra of type $A^{(1)}_{e-1}$*

In this subsection, we follow the paper by Lascoux, Leclerc and Thibon [11]. Let q be an indeterminate over $\mathbb{Q}$, and let $U_q(\widehat{\mathfrak{sl}}_e)$ denote the quantum algebra of type $A^{(1)}_{e-1}$. This is the associative $\mathbb{Q}(q)$-algebra with Chevalley generators $e_i, f_i, t_i^{\pm 1}$ for $0 \leqslant i \leqslant e-1$, subject to relations which may be found in [11].

Define the *Fock space of type* $A^{(1)}_{e-1}$ to be the $\mathbb{Q}(q)$-vector space $\mathcal{F}$ with basis consisting of symbols $|\lambda\rangle$ for all partitions λ. This has the structure of a module for $U_q(\widehat{\mathfrak{sl}}_e)$. A full description of the module action may be found in [11]; all we need is the action of the generators $f_0, \ldots, f_{e-1}$

and their divided powers $f_i^{(m)} = \frac{f_i^m}{[m]_q!}$, which we now describe. If ν, λ are any partitions, then we write $\nu \xrightarrow{i:m} \lambda$ to indicate that $\nu \subseteq \lambda$ and that $\lambda \setminus \nu$ consists of m nodes of residue i. If $\nu \xrightarrow{i:m} \lambda$ and $\mathfrak{x}$ is any node of $\lambda \setminus \nu$, then we set

$$\begin{aligned} N_{\mathfrak{x}}(\nu,\lambda) = {} & (\text{number of addable nodes of } \lambda \text{ of residue } i \text{ to the left of } \mathfrak{x}) \\ & - (\text{number of removable nodes of } \nu \text{ of residue } i \text{ to the left of } \mathfrak{x}) \end{aligned}$$

and

$$N(\nu,\lambda) = \sum_{\mathfrak{x} \in \lambda \setminus \nu} N_{\mathfrak{x}}(\nu,\lambda).$$

Then we have the following.

Suppose ν is any partition, $m \geqslant 1$ and $0 \leqslant i \leqslant e-1$. Then

$$f_i^{(m)}|\nu\rangle = \sum_{\lambda \xleftarrow{i:m} \nu} q^{N(\nu,\lambda)}|\lambda\rangle.$$

The submodule of $\mathcal{F}$ generated by the vector $|\varnothing\rangle$ is a realisation of the basic representation $M(\Lambda_0)$ of $U_q(\widehat{\mathfrak{sl}}_e)$. This possesses a *canonical basis*, which consists of vectors

$$G(\mu) = \sum_{\lambda} d_{\lambda\mu}(q)|\lambda\rangle$$

for all e-restricted partitions μ. The coefficients $d_{\lambda\mu}(q)$ are called *q-decomposition numbers*, and have the following remarkable properties:

(1) $d_{\lambda\mu}(q)$ is a polynomial in q with non-negative integer coefficients;
(2) $d_{\lambda\mu}(q) = 0$ unless λ and μ are partitions of the same integer and $\mu \trianglelefteqslant \lambda$;
(3) $d_{\mu\mu}(q) = 1$, while if $\mu \triangleleft \lambda$ then $d_{\lambda\mu}$ is divisible by q;
(4) (Ariki's Theorem [1, Theorem 4.4]) if v is a primitive eth root of unity in $\mathbb{C}$, and S^λ and D_μ are the Specht module and the simple module for the Iwahori–Hecke algebra $\mathcal{H}_{\mathbb{C},v}(\mathfrak{S}_n)$, then the decomposition number $[S^\lambda : D_\mu]$ equals $d_{\lambda\mu}(1)$.

The canonical basis may be computed recursively using the *LLT algorithm* [11]; we use a slight variation of this, which we now describe. Suppose μ is an e-restricted partition of r, and as above let μ^- be the partition obtained by removing the outermost ladder from μ. Suppose that this ladder consists of m nodes of residue i, and suppose that we have constructed the canonical basis vector $G(\mu^-)$. Suppose also that we have constructed the canonical basis vectors $G(\xi)$ for all e-regular partitions ξ of n such that $\mu \triangleleft \xi$.

(1) Begin by calculating the vector $\hat{G}(\mu) = f_i^{(m)} G(\mu^-)$. Write $\hat{G}(\mu) = \sum_\lambda \hat{d}_{\lambda\mu}(q)|\lambda\rangle$. The coefficients $\hat{d}_{\lambda\mu}(q)$ will be Laurent polynomials with integer coefficients, with $\hat{d}_{\mu\mu}(q) = 1$ and with $d_{\lambda\mu}(q) = 0$ unless λ is a partition of n with $\mu \trianglelefteqslant \lambda$.

(2) If the coefficients $\hat{d}_{\lambda\mu}(q)$ for $\mu \rhd \lambda$ are polynomials divisible by q, then we have $G(\mu) = \hat{G}(\mu)$.
(3) If not, we let ξ be a minimal partition (in the dominance order) such that $\mu \lhd \xi$ and $\hat{d}_{\xi\mu}(q)$ is not a polynomial divisible by q; it happens that ξ is an e-restricted partition of r. There is a unique $h(q) \in \mathbb{Z}[q+q^{-1}]$ such that $\hat{d}_{\xi\mu}(q) - h(q)$ is a polynomial divisible by q; we replace $\hat{G}(\mu)$ with $\hat{G}(\mu) - h(q)G(\xi)$, and return to (2).

2.4. *The main theorem*

In order to state our main theorem, we need to recall some more combinatorial notions and introduce some notation. If ν is a partition and (j,k) is a node of ν, then we define the (j,k)-*hook* $h_\nu(j,k)$ of ν to be the set of nodes of ν which lie directly to the right of (j,k) or directly below (j,k), including (j,k) itself. The rightmost node (j,ν_j) of $h_\nu(j,k)$ is called the *hand node* of $h_\nu(j,k)$, and the bottommost node (ν'_k,k) is the *foot node*. The distance $\nu_j - k$ is the *arm length* of $h_\nu(j,k)$, and the distance $\nu'_k - j$ is the *leg length*. The total number of nodes in $h_\nu(j,k)$ (that is, the arm length plus the leg length plus one) is called the *length* of $h_\nu(j,k)$. We say that the hook $h_\nu(j,k)$ is *shallow* if its length is divisible by e and its foot node lies in an earlier ladder than its hand node. Equivalently, $h_\nu(j,k)$ is shallow if its length is divisible by e and its arm length is more than $e-1$ times its leg length. We write $s(\nu)$ for the number of shallow hooks of ν. Our main theorem is as follows.

Theorem 2.2. *Suppose λ and μ are partitions of n, with μ e-restricted. Then $d_{\lambda\mu}(q) = 0$ if $\mu \ntrianglelefteq \lambda^{\mathrm{R}}$, while $d_{\lambda\lambda^{\mathrm{R}}}(q) = q^{s(\lambda)}$.*

Note that using Ariki's Theorem, Theorem 2.1 and the fact that the polynomials $d_{\lambda\mu}(q)$ have non-negative coefficients, we can see that $d_{\lambda\mu}(q) = 0$ if $\mu \ntrianglelefteq \lambda^{\mathrm{R}}$, while $d_{\lambda\lambda^{\mathrm{R}}}(q)$ is a power of q. The point of this paper is that we find this power of q, and we prove our result without assuming Ariki's Theorem.

We now proceed with the proof. This essentially involves applying the LLT algorithm and examining how the operators $f_i^{(m)}$ affect the number $s(\lambda)$. The following lemma is simply a re-writing of [10, 6.3.54, 6.3.55] with our combinatorial conventions.

Lemma 2.3. *Suppose that λ and μ are partitions of r with μ e-restricted, and that the outermost ladder of μ consists of m nodes of residue i. Suppose that ν is a partition of $r-m$ such that $\nu \xrightarrow{i:m} \lambda$ and $\mu^- \trianglelefteq \nu^{\mathrm{R}}$. Then $\mu \trianglelefteq \lambda^{\mathrm{R}}$, with equality only if $\mu^- = \nu^{\mathrm{R}}$.*

Proof. Let $l = \mu'_1$. Since μ is e-restricted and $\mu \setminus \mu^-$ consists of the nodes in the outermost ladder of μ, these nodes lie in rows $l-m+1, l-m+2, \dots, l$. So for $j \leqslant l$ we have

$$\mu_1 + \dots + \mu_j = \mu_1^- + \dots + \mu_j^- + \max\{0, m-l+j\}.$$

Since the number of nodes in each ladder of ν is at most the number of nodes in the corresponding ladder of λ, we have $\nu^{\mathrm{R}} \subseteq \lambda^{\mathrm{R}}$, and hence $\nu^{\mathrm{R}} \xrightarrow{i:m} \lambda^{\mathrm{R}}$. We claim that $\lambda_1^{\mathrm{R}\prime} \leqslant l$. We certainly have $\nu_1^{\mathrm{R}\prime} \leqslant \mu_1^{-\prime} \leqslant \mu'_1 = l$, so the only way we can have $\lambda_1^{\mathrm{R}\prime} > l$ is if the node $(l+1,1)$ is a node of $\lambda^{\mathrm{R}} \setminus \nu^{\mathrm{R}}$. In particular, this means that the node $(l+1,1)$ has residue i. But then the smallest k for which the node (l,k) has residue i is $k=e$; since μ has a node of residue i in row l, this means that $\mu_l \geqslant e$, so that μ is not e-restricted; contradiction.

So λ^{R} is a partition with at most l non-zero parts, obtained by adding m nodes of residue i to ν^{R}. There can be at most one of these nodes in any row (and none below row l), so for any $j \leqslant l$ we have

$$\lambda_1^{\mathrm{R}} + \cdots + \lambda_j^{\mathrm{R}} \geqslant \nu_1^{\mathrm{R}} + \cdots + \nu_j^{\mathrm{R}} + \max\{0, m-l+j\}.$$

Hence for $j \leqslant l$ we have

$$\begin{aligned}\mu_1 + \cdots + \mu_j &= \mu_1^- + \cdots + \mu_j^- + \max\{0, m-l+j\} \\ &\leqslant \nu_1^{\mathrm{R}} + \cdots + \nu_j^{\mathrm{R}} + \max\{0, m-l+j\} \\ &\leqslant \lambda_1^{\mathrm{R}} + \cdots + \lambda_j^{\mathrm{R}},\end{aligned}$$

while for $j \geqslant l$ we have

$$\mu_1 + \cdots + \mu_j = \lambda_1^{\mathrm{R}} + \cdots + \lambda_j^{\mathrm{R}} = r.$$

So $\mu \trianglelefteq \lambda^{\mathrm{R}}$. In order to get $\mu = \lambda^{\mathrm{R}}$, we should need equality everywhere; in particular, $\mu_1^- + \cdots + \mu_j^- = \nu_1^{\mathrm{R}} + \cdots + \nu_j^{\mathrm{R}}$ for each j, i.e. $\mu^- = \nu^{\mathrm{R}}$. $\square$

Lemma 2.4. *Suppose $h_\nu(j,k)$ is a shallow hook of a partition ν. Then the node $(k, \nu'_k + 1)$ below the foot node of $h_\nu(j,k)$ lies in the same ladder as, or an earlier ladder than, the hand node of $h_\nu(j,k)$.*

Proof. Suppose the foot node of $h_\nu(j,k)$ lies in ladder l, and the hand node lies in ladder m. Then $l < m$, and the node $(k, \nu'_k + 1)$ lies in ladder $l + e - 1$. The fact that the length of $h_\nu(j,k)$ is divisible by e implies that this node has the same residue as the foot node of $h_\nu(j,k)$, so we have $l + e - 1 \equiv m \pmod{e}$. We deduce that $l + e - 1 \leqslant m$. $\square$

Proposition 2.5. *Suppose that λ is a partition of r, and that the outermost ladder of λ consists of m nodes of residue i. Then the coefficient of λ in $f_i^{(m)}|\lambda^-\rangle$ is $q^{s(\lambda)-s(\lambda^-)}$.*

Proof. For this proof, we write $\mathrm{col}(\mathfrak{x})$ for the column in which a node $\mathfrak{x}$ lies; that is, $\mathrm{col}(j,k) = k$. We need to examine the relationship between the Young diagrams of λ and λ^-, and find all nodes (j,k) for which exactly one of $h_\lambda(j,k)$ and $h_{\lambda^-}(j,k)$ is shallow. Let X be the set of nodes in the outermost ladder of λ, and consider a node (j,k) of λ. Let $\mathfrak{h}$ and $\mathfrak{f}$ denote the hand node and the foot node of $h_\lambda(j,k)$, respectively. We consider whether $\mathfrak{h}$ and/or $\mathfrak{f}$ lie(s) in X.

Suppose $\mathfrak{f}$ lies in X. Since X consists of the nodes in the outermost ladder of λ, $\mathfrak{h}$ must lie in the same ladder as $\mathfrak{f}$ or an earlier ladder than $\mathfrak{f}$. So $h_\lambda(j,k)$ is not shallow. Furthermore, the hand node of $h_{\lambda^-}(j,k)$ lies in a strictly earlier ladder than $\mathfrak{f}$, since if $\mathfrak{h}$ lies in the same ladder as $\mathfrak{f}$, then $\mathfrak{h} \in X$, in which case the hand node of $h_{\lambda^-}(j,k)$ is the node immediately to the left of $\mathfrak{h}$. $\mathfrak{f}$ is the node immediately below the foot node of $h_{\lambda^-}(j,k)$, so $h_{\lambda^-}(j,k)$ is not shallow, by Lemma 2.4.

So we need only consider the case where $\mathfrak{h}$ lies in X but $\mathfrak{f}$ does not (clearly if neither $\mathfrak{h}$ nor $\mathfrak{f}$ lies in X, then $h_\lambda(j,k)$ is shallow if and only if $h_{\lambda^-}(j,k)$ is shallow). In this case, $h_\lambda(j,k)$ is shallow if and only if its length is divisible by e, which happens if and only if the node $(k, \lambda'_k + 1)$ below $\mathfrak{f}$ has residue i. The hand node of $h_{\lambda^-}(j,k)$ is the node immediately to the left of $\mathfrak{h}$, and so lies in the outermost ladder of λ^-. Hence $h_{\lambda^-}(j,k)$ is shallow if and only if its length is divisible by e, which happens if and only if the node $(k, \lambda_k^{-\prime})$ has residue i.

So we find that

$$s(\lambda) - s(\lambda^-) = \left|\left\{(k, \mathfrak{x}) \mid \mathfrak{x} \in X,\ 1 \leqslant k < \operatorname{col}(\mathfrak{x}),\ \operatorname{res}\left(k, \lambda_k' + 1\right) = i\right\}\right| \\ - \left|\left\{(k, \mathfrak{x}) \mid \mathfrak{x} \in X,\ 1 \leqslant k < \operatorname{col}(\mathfrak{x}),\ \operatorname{res}\left(k, \lambda_k^{-\prime}\right) = i\right\}\right|.$$

If $\operatorname{res}(k, \lambda_k' + 1) = i$, then either $(k, \lambda_k' + 1)$ is an addable node of λ of residue i, or $k > 1$ and $\lambda_k' = \lambda_{k-1}'$. Similarly, if $\operatorname{res}(k, \lambda_k^{-\prime}) = i$, then either $(k, \lambda_k^{-\prime})$ is a removable node of λ^- of residue i, or $\lambda_k^{-\prime} = \lambda_{k+1}^{-\prime}$. So we have

$$\begin{aligned} s(\lambda) - s(\lambda^-) = \sum_{\mathfrak{x} \in X} \Big(& (\text{number of addable nodes of } \lambda \text{ of residue } i \text{ to the left of } \mathfrak{x}) \\ & - (\text{number of removable nodes of } \lambda^- \text{ of residue } i \text{ to the left of } \mathfrak{x}) \\ & + \left|\left\{1 < k < \operatorname{col}(\mathfrak{x}) \mid \operatorname{res}\left(k, \lambda_k' + 1\right) = i,\ \lambda_k' = \lambda_{k-1}'\right\}\right| \\ & - \left|\left\{k < \operatorname{col}(\mathfrak{x}) \mid \operatorname{res}\left(k, \lambda_k^{-\prime}\right) = i,\ \lambda_k^{-\prime} = \lambda_{k+1}^{-\prime}\right\}\right| \Big). \end{aligned}$$

Now if $\operatorname{res}(k, \lambda_k' + 1) = i$ and $\lambda_k' = \lambda_{k-1}'$, then $\lambda_{k-1}^{-\prime} = \lambda_k^{-\prime} = \lambda_k'$; furthermore, if $\operatorname{res}(k, \lambda_k^{-\prime}) = i$ and $\lambda_k^- = \lambda_{k+1}^{-\prime}$, then $k + 1 < \operatorname{col}(\mathfrak{x})$. Hence the last two terms in the above sum cancel, and so we get

$$s(\lambda) - s(\lambda^-) = N(\lambda^-, \lambda)$$

as required. □

Proof of Theorem 2.2. We proceed by induction on r. The case $r = 0$ is trivial, so we suppose that μ is an e-restricted partition of $r > 1$, and that the result holds when μ is replaced with μ^-. We also use induction on the dominance order, and assume that the result holds when μ is replaced with any e-restricted partition ξ such that $\mu \lhd \xi$.

Suppose that the outermost ladder of μ consists of m nodes of residue i, and consider $f_i^{(m)}(G(\mu^-)) = \sum_\lambda \hat{d}_{\lambda\mu}(q)|\lambda\rangle$. First we prove that

$$\hat{d}_{\lambda\mu}(q) = \begin{cases} q^{s(\lambda)} & (\mu = \lambda^{\mathrm{R}}), \\ 0 & (\mu \not\trianglelefteq \lambda^{\mathrm{R}}). \end{cases}$$

If $\hat{d}_{\lambda\mu}(q) \neq 0$, then there is a partition ν of $n - m$ such that $d_{\nu\mu^-}(q) \neq 0$ and $\nu \xrightarrow{i:m} \lambda$. By induction, we have $\mu^- \trianglelefteq \nu^{\mathrm{R}}$, and so by Lemma 2.3 we have $\mu \trianglelefteq \lambda^{\mathrm{R}}$, which gives the second part of the claim. Now suppose $\mu = \lambda^{\mathrm{R}}$. If ν is a partition of $n - m$ such that $d_{\nu\mu^-}(q) \neq 0$ and $\nu \xrightarrow{i:m} \lambda$, then by induction and by the last part of Lemma 2.3 we must have $\mu^- = \nu^{\mathrm{R}}$, which means that $\nu = \lambda^-$. So $\hat{d}_{\lambda\mu}(q)$ equals $d_{\lambda^-(\lambda^-)^{\mathrm{R}}}(q)$ times the coefficient of $|\lambda\rangle$ in $f_i^{(l)}\lambda^-$; by induction and by Proposition 2.5, this is $q^{s(\lambda^-)}.q^{s(\lambda)-s(\lambda^-)} = q^{s(\lambda)}$.

So we have proved our result for the Laurent polynomials $\hat{d}_{\lambda\mu}(q)$. By the LLT algorithm we have

$$d_{\lambda\mu}(q) = \hat{d}_{\lambda\mu}(q) + \sum_{\xi \rhd \mu} c_{\mu\xi}(q) d_{\lambda\xi}(q)$$

for some coefficients $c_{\mu\xi}(q)$. Now if $\mu \not\trianglelefteq \lambda^{\mathrm{R}}$, then for any ξ with $\xi \rhd \mu$ we have $\xi \not\trianglelefteq \lambda^{\mathrm{R}}$ and so (by our inductive assumption) $d_{\lambda\xi}(q) = 0$; so $d_{\lambda\mu}(q) = \hat{d}_{\lambda\mu}(q)$, and the result follows. $\square$

3. The Fock space of type $A^{(2)}_{2n}$

For this section, we fix a positive integer n, and undertake the same task as in Section 2 for the Fock space of type $A^{(2)}_{2n}$. This Fock space has combinatorics closely related to the theory of projective (or spin) representations of symmetric groups, and there are conjectural links between the canonical basis coefficients and spin representations. Assuming such conjectures hold, our main theorem is a q-analogue of a recent theorem of Brundan and Kleshchev, which itself is an analogue of Theorem 2.1 for spin representations. The method of proof of our theorem is the same as for Theorem 2.2, but the combinatorial technicalities are more complicated.

3.1. Residues, ladders and h-restriction

Put $h = 2n + 1$. We say that a partition λ is *h-strict* if $\lambda_j > \lambda_{j+1}$ whenever h does not divide λ_j, and we write DP_h for the set of h-strict partitions. We have $\mathrm{DP}_h = \bigcup_{r \geqslant 0} \mathrm{DP}_h(r)$, where $\mathrm{DP}_h(r)$ denotes the set of h-strict partitions of r. In this section we shall only be concerned with h-strict partitions. We say that an h-strict partition is *restricted* if for every j we have either $\lambda_j - \lambda_{j+1} = h$ and $h \nmid \lambda_j$, or $\lambda_j - \lambda_{j+1} < h$.

We now need to re-define some combinatorial notions. We mostly follow [12]; some of the conventions therefore differ from those in [3,4]. We define the *residue* of a node $\mathfrak{x} = (j, k)$ to be the integer i such that $|i| \leqslant n$ and $i \equiv k + n \pmod{h}$. Note that the residue of a node depends only on the column in which it lies; accordingly, we can define the residue of any column of a Young diagram in the obvious way.

Given any $l \geqslant 1$, we define the lth *ladder* in $\mathbb{N} \times \mathbb{N}$ to be the set

$$\left\{(j,k) \in \mathbb{N} \times \mathbb{N} \;\middle|\; \left\lfloor \frac{2nk}{2n+1} \right\rfloor + 2nj - 2n + 1 = l \right\}.$$

Note that if $l \not\equiv 1 \pmod{2n}$, then the nodes in ladder l all have the same residue, while if $l \equiv 1 \pmod{2n}$, then the nodes in ladder l have residue $\pm n$.

If $\lambda \in \mathrm{DP}_h$, the lth ladder of λ is the set of nodes of λ lying in the lth ladder of $\mathbb{N} \times \mathbb{N}$. We define the *outermost* ladder of λ as before, and let λ^- be the partition obtained by removing all the nodes in the outermost ladder of λ. It is very easy to see that $\lambda^- \in \mathrm{DP}_h$. It is also easy to see that λ is restricted if and only if for each l the nodes in the lth ladder of λ are as far to the left as possible. For any $\lambda \in \mathrm{DP}_h$, we define its *$h$-restriction* by moving all the nodes in each ladder as far to the left in that ladder as possible. As with e-restriction in the previous section, it is an easy exercise to show that we obtain the Young diagram of a restricted h-strict partition, which we write as λ^{R}, in this way.

Example. Suppose $n = 2$, and $\lambda = (10, 10, 2) \in \mathrm{DP}_5$. Then $\lambda^{\mathrm{R}} = (10, 7, 4, 1)$, as we can see from the following Young diagrams, in which we label each node according to the ladder in which it lies:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&5&6&7&8&9\\\hline 5&6&7&8&9&9&10&11&12&13\\\hline 9&10\\\cline{1-2}\end{array}\,, \qquad \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&5&6&7&8&9\\\hline 5&6&7&8&9&9&10\\\cline{1-7} 9&10&11&12\\\cline{1-4} 13\\\cline{1-1}\end{array}\,.$$

3.2. *Spin representations of symmetric groups*

Suppose $\mathbb{F}$ is a field of odd characteristic p, and let $n = \frac{p-1}{2}$. For any r, let $\mathcal{T}_r$ denote the non-trivially twisted group algebra of the symmetric group $\mathfrak{S}_r$. The representation theory of $\mathcal{T}_n$ describes the *projective* or *spin* representations of $\mathfrak{S}_r$ over $\mathbb{F}$, or equivalently the representations of a Schur cover of $\mathfrak{S}_r$. The theory of projective representations of the symmetric group is well studied; the references [2,3,15] provide summaries at various stages in the development of the theory.

$\mathcal{T}_r$ is most conveniently viewed as a superalgebra. For each p-strict partition λ of r, Brundan and Kleshchev define a 'Specht supermodule' $S(\lambda)$ (which is actually a virtual supermodule); in the case where λ is *strict* (that is, $\lambda_j > \lambda_{j+1}$ whenever $\lambda_j > 0$) this Specht supermodule arises as the p-modular reduction of a Specht supermodule over a field of infinite characteristic. Computing the composition factors of the Specht supermodules is therefore equivalent (modulo the transition from supermodules to modules) to the decomposition number problem for spin representations of symmetric groups. If λ is a restricted p-strict partition of r, then there is a simple supermodule $D(\lambda)$, and the supermodules $D(\lambda)$ give all the simple supermodules of $\mathcal{T}_r$ as λ ranges over the set of restricted p-strict partitions of r.

Brundan and Kleshchev have proved the following theorem, which is an analogue for spin representations of Theorem 2.1.

Theorem 3.1. *(See [4, Theorem 1.2(i)].) Suppose λ and μ are p-strict partitions of r, with μ restricted. Then $[S(\lambda) : D(\mu)] = 0$ unless $\mu \trianglelefteq \lambda^{\mathrm{R}}$, while $[S(\lambda) : D(\lambda^{\mathrm{R}})] = 1$.*

Our main theorem is to some extent a q-analogue of this theorem, as we shall see in the following section.

3.3. *The quantum algebra of type $A^{(2)}_{2n}$*

Let $U_q(A^{(2)}_{2n})$ denote the quantum algebra of type $A^{(2)}_{2n}$, with Chevalley generators $e_i, f_i, t_i^{\pm 1}$ for $0 \leqslant i \leqslant n$. Define the *Fock space of type $A^{(2)}_{2n}$* to be the $\mathbb{Q}(q)$-vector space $\mathcal{F}$ with basis consisting of symbols $|\lambda\rangle$ for all partitions $\lambda \in \mathrm{DP}_h$. This has the structure of a module for $U_q(A^{(2)}_{2n})$; the description of the module action is slightly complicated, and may be found in [12]; later, we shall describe particular cases of the action of the generators $f_0, \dots, f_n$ and their divided powers $f_0^{(m)}, \dots, f_n^{(m)}$ (whose definition is itself not straightforward).

The submodule $\mathcal{F}$ generated by $|\varnothing\rangle$ is a realisation of the basic representation $M(\Lambda_n)$ of $U_q(A^{(2)}_{2n})$. It has a canonical basis consisting of vectors

$$G(\mu) = \sum_{\lambda \in \mathrm{DP}_h} d_{\lambda\mu}(q)|\lambda\rangle$$

for all restricted h-strict partitions μ. The coefficients $d_{\lambda\mu}(q)$ satisfy similar properties to the q-decomposition numbers in type $A^{(1)}$, and we call $d_{\lambda\mu}(q)$ a q-decomposition number here too, in view of the conjecture by Leclerc and Thibon [12, Conjecture 6.2] that if h is a prime which is sufficiently large compared to $|\mu|$, then $d_{\lambda\mu}(1)$ equals the decomposition number $[S(\lambda) : D(\mu)]$.

The canonical basis may be computed using the algorithm described in [12]; this is exactly the same as the LLT algorithm, but with the appropriate notion of ladders.

3.4. The main theorem

In order to state our main result, we introduce a function similar to the function $s(\lambda)$ introduced in the last section. Suppose $\lambda \in \mathrm{DP}_h$. For a positive integer a, we say that λ has a *shallow ah-bar* in row j if

- $\lambda_{j+a} < \lambda_j - ah$, and
- either $h \mid \lambda_j$ or there is no l such that $\lambda_l = \lambda_j - ah$.

We say that λ has a *semi-shallow ah-bar* in row j if

- $\lambda_{j+a} = \lambda_j - ah$, and
- $h \mid \lambda_j$.

We define the *number of (semi-)shallow bars of* λ to be the number of pairs (a, j) such that λ has a (semi-)shallow ah-bar in row j.

Finally, we say that λ has a *shallow h-pair* in rows j and l (for $j < l$) if

- $h \nmid \lambda_j$,
- $h \mid \lambda_j + \lambda_l$, and
- $\lambda_j - \lambda_l > h(l - j)$.

Now define

$$\begin{aligned} t(\lambda) = 2 \times (&\text{number of shallow bars of } \lambda) \\ &+ (\text{number of semi-shallow bars of } \lambda) \\ &+ 2 \times (\text{number of shallow } h\text{-pairs of } \lambda). \end{aligned}$$

Our main theorem is as follows.

Theorem 3.2. *Suppose $\lambda, \mu \in \mathrm{DP}_h$, with μ restricted. Then $d_{\lambda\mu}(q) = 0$ if $\mu \ntrianglelefteq \lambda^{\mathrm{R}}$, while $d_{\lambda\lambda^{\mathrm{R}}}(q) = q^{t(\lambda)}$.*

We prove Theorem 3.2 in much the same way as we proved Theorem 2.2. In order to state counterparts of Lemma 2.3 and Proposition 2.5, we need to introduce the q-divided powers of the Chevalley generators $f_0, \dots, f_n$. For $0 \leqslant i \leqslant n$ and $m \geqslant 1$, we define

$$f_k^{(m)} = \frac{f_k^m}{[m]_{q_k}!},$$

where

$$q_k = \begin{cases} q^4 & (k = 0), \\ q^2 & (1 \leqslant k \leqslant n-1), \\ q & (k = n). \end{cases}$$

As noted above, it is awkward to describe the action of these divided powers on the Fock space, and for the case $k = n$ we shall give only a partial description. A full description is given in Section 2 of [12], where the vector $|\lambda\rangle$ is written as $u_{\lambda_1} \wedge u_{\lambda_2} \wedge \cdots$, and the action of the f_i is given by Eqs. (7), (8), (11)–(13) and (17), together with the commutation rules (Eqs. (14) and (15)). An easy consequence of these definitions is the following result, in which for $\lambda, \nu \in \mathrm{DP}_h$ we write $\nu \xrightarrow{\pm k:m} \lambda$ to indicate that $\nu \subseteq \lambda$ and that $\lambda \setminus \nu$ consists of m nodes of residue $\pm k$.

Lemma 3.3. *If $\nu \in \mathrm{DP}_h$, then $f_k^{(m)}|\nu\rangle$ is a linear combination of vectors $|\lambda\rangle$ for which $\nu \xrightarrow{\pm k:m} \lambda$.*

Our analogues of Lemma 2.3 and Proposition 2.5 are as follows.

Lemma 3.4. *Suppose that $\lambda, \mu \in \mathrm{DP}_h(r)$ with μ restricted. Suppose that the outermost ladder of μ consists of m nodes of residue $\pm k$. Suppose that $\nu \in \mathrm{DP}_h(r-m)$ satisfies $\nu \xrightarrow{\pm k:m} \lambda$ and $\mu^- \trianglelefteqslant \nu^{\mathrm{R}}$. Then $\mu \trianglelefteqslant \lambda^{\mathrm{R}}$, with equality only if $\mu^- = \nu^{\mathrm{R}}$.*

Proposition 3.5. *Suppose that $\lambda \in \mathrm{DP}_h$, and that the outermost ladder of λ consists of m nodes of residue $\pm k$, for $0 \leqslant k \leqslant n$. Then the coefficient of $|\lambda\rangle$ in $f_k^{(m)}|\lambda^-\rangle$ is $q^{t(\lambda)-t(\lambda^-)}$.*

Given these results, the proof of Theorem 3.2 may be completed in exactly the same way as Theorem 2.2, using the 'LT algorithm' [12, §4] rather than the LLT algorithm. So it suffices to prove Lemma 3.4 and Proposition 3.5, which is the purpose of the remainder of this paper.

Proof of Lemma 3.4. For the case $k < n$, this is essentially identical to the proof of Lemma 2.3; for the case $k = n$, we need to make some slight adjustments. Again, we put $l = \mu'_1$. Now $\mu \setminus \mu^-$ consists of one node in row l, two nodes in each of rows $l - \lfloor\frac{m-1}{2}\rfloor, l - \lfloor\frac{m-1}{2}\rfloor + 1, \dots, l-1$, and one node in row $l - \frac{m}{2}$ if m is even. Hence for $j < l$ we have

$$\mu_1 + \cdots + \mu_j = \mu_1^- + \cdots + \mu_j^- + \max\{0, m - 2l + 2j + 1\}.$$

Now the node of $\mu \setminus \mu^-$ in row l must be the node $(l, 1)$, so $\mu_1^{-\prime} = l - 1$. Hence $\nu_1^{\mathrm{R}\prime} \leqslant l - 1$, so that $\lambda_l^{\mathrm{R}} \leqslant 1$ and $\lambda_{l+1}^{\mathrm{R}} = 0$. As in the proof of Lemma 2.3, $\lambda^{\mathrm{R}} \setminus \nu^{\mathrm{R}}$ consists of m nodes of residue $\pm n$; there can be no more than two of these in any row, no more than one in row l, and none below row l. So for any $j < l$ we have

$$\lambda_1^{\mathrm{R}} + \cdots + \lambda_j^{\mathrm{R}} \geqslant \nu_1^{\mathrm{R}} + \cdots + \nu_j^{\mathrm{R}} + \max\{0, m - 2l + 2j + 1\}.$$

Hence

$$\begin{aligned}\mu_1 + \cdots + \mu_j &= \mu_1^- + \cdots + \mu_j^- + \max\{0, m - 2l + 2j + 1\}\\ &\leqslant \nu_1^{\mathrm{R}} + \cdots + \nu_j^{\mathrm{R}} + \max\{0, m - 2l + 2j + 1\}\\ &\leqslant \lambda_1^{\mathrm{R}} + \cdots + \lambda_j^{\mathrm{R}}\end{aligned}$$

for $j < l$, while for $j \geqslant l$ we have

$$\mu_1 + \cdots + \mu_j = \lambda_1^{\mathrm{R}} + \cdots + \lambda_j^{\mathrm{R}} = r.$$

So we have $\mu \trianglelefteq \lambda^{\mathrm{R}}$. To get $\mu = \lambda^{\mathrm{R}}$, we must have $\mu_1^- + \cdots + \mu_j^- = \nu_1^{\mathrm{R}} + \cdots + \nu_j^{\mathrm{R}}$ for every j, i.e. $\nu^- = \nu^{\mathrm{R}}$. $\square$

Now we prove Proposition 3.5, beginning with the case where $k < n$. Here we can give a complete description of the action of $f_k^{(m)}$. The following is a translation of the definitions in [12, §2] into the terminology of addable and removable nodes.

Proposition 3.6. *Suppose $0 \leqslant k < n$, and that $\lambda, \nu \in \mathrm{DP}_h$ with $\nu \xrightarrow{\pm k:m} \lambda$. For each $\mathfrak{x} \in \lambda \setminus \nu$, define*

$$\begin{aligned}N_{\mathfrak{x}}(\nu, \lambda) = {}&(\textit{number of addable nodes of } \nu \textit{ of residue } k \textit{ to the left of } \mathfrak{x})\\ &- (\textit{number of removable nodes of } \lambda \textit{ of residue } k \textit{ to the left of } \mathfrak{x})\\ &+ (\textit{number of addable nodes of } \lambda \textit{ of residue } -k \textit{ to the left of } \mathfrak{x})\\ &- (\textit{number of removable nodes of } \nu \textit{ of residue } -k \textit{ to the left of } \mathfrak{x})\end{aligned}$$

and $N(\nu, \lambda) = \sum_{\mathfrak{x} \in \lambda \setminus \nu} N_{\mathfrak{x}}(\nu, \lambda)$. Then the coefficient of $|\lambda\rangle$ in $f_k^{(m)}|\nu\rangle$ equals $q^{2N(\nu,\lambda)}$.

Remark. In the definition of $N_{\mathfrak{x}}(\nu, \lambda)$, we could replace 'addable nodes of ν' in the first term and 'removable nodes of λ' in the second term with 'addable nodes of λ' and 'removable nodes of ν' respectively; a similar statement applies to the third and fourth terms. We have used the formulation above because it is most useful in the calculations below. Note that when $k = 0$, some nodes get counted twice—this indicates why the expression for the action of f_0 in [12] differs from the expression for the action of f_i for $0 < i < n$.

Proof of Proposition 3.5 *(case $k < n$).* Let X be the set of nodes in the outermost ladder of λ. Since $k < n$, the nodes in X all have the same residue. Call this residue i; then $|i| = k$. It suffices to prove the following three claims.

Claim 1. *The number of semi-shallow bars of λ equals the number of semi-shallow bars of λ^-.*

Claim 2. *The number of shallow bars of* λ *minus the number of shallow bars of* λ^- *equals*

$$\sum_{\mathfrak{x}\in X}(\textit{number of addable nodes of } \lambda^- \textit{ of residue } i \textit{ to the left of } \mathfrak{x}$$
$$- \textit{number of removable nodes of } \lambda \textit{ of residue } i \textit{ to the left of } \mathfrak{x}).$$

Claim 3. *The number of shallow* h*-pairs of* λ *minus the number of shallow* h*-pairs of* λ^- *equals*

$$\sum_{\mathfrak{x}\in X}(\textit{number of addable nodes of } \lambda \textit{ of residue } -i \textit{ to the left of } \mathfrak{x}$$
$$- \textit{number of removable nodes of } \lambda^- \textit{ of residue } -i \textit{ to the left of } \mathfrak{x}).$$

Claim 1 is straightforward: λ has a semi-shallow ah-bar in row j if and only if

- the node at the end of row j has residue n,
- $(j+a, \lambda_j - ah + 1)$ is not a node of λ, and
- either $\lambda_j = ah$ or $(j+a, \lambda_j - ah)$ is a node of λ;

these conditions cannot be affected by removing nodes whose residue is not $\pm n$.

Now we prove Claim 2. For each pair j, a, we consider whether either λ or λ^- has a shallow ah-bar in row j. Let $\mathfrak{h}$ be the node at the end of row j of λ.

Claim 4. *The following conditions are equivalent.*

(1) λ *has a shallow* ah*-bar in row* j *but* λ^- *does not.*
(2) $\mathfrak{h} \in X$, $\lambda_j > ah$, λ^- *has an addable node in column* $\lambda_j - ah$ *and* λ *does not have a removable node in column* $\lambda_j - ah$.

Proof. (2) $\Rightarrow$ (1). Let $\mathfrak{f} = (j+b, \lambda_j - ah)$ be the node at the bottom of column $\lambda_j - ah$ of λ. Since $\mathfrak{f}$ is not a removable node of λ, $\mathfrak{f}$ cannot lie in X. In particular, $\mathfrak{f}$ lies in an earlier ladder than $\mathfrak{h}$, and so $b < a$. The fact that $\mathfrak{f}$ is not removable also implies that no part of λ equals $\lambda_j - ah$, so λ has a shallow ah-bar in row j.

Since λ^- has an addable node in column $\lambda_j - ah$, we have $\lambda^-_l = \lambda_j - ah - 1 = \lambda^-_j - ah$ for some l. Furthermore, $\lambda^-_j \equiv i + n \not\equiv 0 \pmod h$, and so λ^- does not have a shallow ah-bar in row j.

(1) $\Rightarrow$ (2). First suppose that $\mathfrak{h} \notin X$. Then we have $\lambda^-_j - ah = \lambda_j - ah > \lambda_{j+a} \geqslant \lambda^-_{j+a}$. Since λ^- does not have a shallow ah-bar in row j, we must therefore have $h \nmid \lambda^-_j$ and $\lambda^-_l = \lambda^-_j - ah$ for some l. We have $l < j+a$, so the node (l, λ^-_l) lies in earlier ladder than $\mathfrak{h}$. In particular, this node does not lie in X, so that $\lambda_l = \lambda^-_l = \lambda_j - ah$, contradicting the fact that λ has a shallow ah-bar in row j.

So we must have $\mathfrak{h} \in X$. Since $i \neq n$, h does not divide λ_j, so λ cannot have any part equal to $\lambda_j - ah$. So $\lambda_j > ah$ and there is no removable node in column $\lambda_j - ah$ of λ. If there is no addable node in column $\lambda_j - ah$ of λ^-, then there is no l with $\lambda^-_l = \lambda_j - ah - 1 = \lambda^-_j - ah$. Furthermore, we have $\lambda^-_{j+a} \leqslant \lambda_{j+a} \leqslant \lambda_j - ah - 1 = \lambda^-_j - ah$, so that λ^- has a shallow ah-bar in row j; contradiction. So there must be an addable node in column $\lambda_j - ah$ of λ. $\square$

The proof of the following claim is very similar.

Claim 5. *The following are equivalent.*

(1) λ^- *has a shallow* ah*-bar in row* j *but* λ *does not.*
(2) $\mathfrak{h} \in X$, $\lambda_j > ah$, λ *has a removable node in column* $\lambda_j - ah$*, and* λ^- *does not have an addable node in column* $\lambda_j - ah$.

Using these two claims for all a, j, we find that the number of shallow bars of λ minus the number of shallow bars of λ^- equals

$$\sum_{\mathfrak{x}\in X}(\text{number of addable nodes of } \lambda^- \text{ to the left of } \mathfrak{x} \text{ of residue } i$$
$$- \text{ number of removable nodes of } \lambda \text{ to the left of } \mathfrak{x} \text{ of residue } i),$$

as required.

Now we move on to Claim 3. Suppose $j < l$ and $\lambda_l > 0$, and let $\mathfrak{h}$ and $\mathfrak{f}$ be the nodes at the ends of rows j and l of λ, respectively.

Claim 6. *The following conditions are equivalent.*

(1) λ *has a shallow* h*-pair in rows* j *and* l *but* λ^- *does not.*
(2) $\mathfrak{h} \in X$ *and there is an addable node of residue* $-i$ *in row* l *of* λ.

Proof. (2) $\Rightarrow$ (1). Since $\operatorname{res}(\mathfrak{h}) = i \neq n$, we have $h \nmid \lambda_j$. We also have $\operatorname{res}(l, \lambda_l + 1) = -i$, so $h \mid \lambda_j + \lambda_l$. The residue of $\mathfrak{f}$ is $-i - 1 \neq i$, so $\mathfrak{f}$ must lie in an earlier ladder than $\mathfrak{h}$, and hence $\lambda_j - \lambda_l > h(l - j)$. So λ has a shallow h-pair in rows j and l.

Now $\lambda^-_j = \lambda_j - 1$ and $\lambda^-_l = \lambda_l$, since $\mathfrak{f} \notin X$. Hence $h \nmid \lambda^-_j + \lambda^-_l$, so λ^- does not have a shallow h-pair in rows j and l.

(1) $\Rightarrow$ (2). $\mathfrak{h}$ and $\mathfrak{f}$ cannot both lie in λ^- (since if they did, then λ^- would have a shallow h-pair in rows j and l if and only if λ did), so at least one of them lies in X. But the fact that λ has a shallow h-pair implies that $\mathfrak{f}$ lies in a strictly earlier ladder than $\mathfrak{h}$, so $\mathfrak{h}$ must lie in X, and in particular has residue i. The fact that $h \mid \lambda_j + \lambda_l$ implies that the node immediately to the right of $\mathfrak{f}$ has residue $-i$; this node must be an addable node of λ, since $\lambda_l < \lambda_{l-1}$. □

In a very similar way, we prove the following.

Claim 7. *The following conditions are equivalent.*

(1) λ^- *has a shallow* h*-pair in rows* j *and* l *but* λ *does not.*
(2) $\mathfrak{h} \in X$ *and there is a removable node of residue* $-i$ *in row* l *of* λ^-.

Claims 6 and 7 immediately imply Claim 3. □

The proof for the case where the outermost ladder of λ consists of nodes of residue $\pm n$ is rather more complicated. To begin with, we must describe the coefficient of $|\lambda\rangle$ in $f_n^{(m)}|\nu\rangle$, given certain conditions on λ and ν. We begin with the case $m = 1$.

Lemma 3.7. *Suppose that $\lambda, \nu \in \mathrm{DP}_h$ with $\nu \xrightarrow{\pm n:1} \lambda$, and let $\mathfrak{x} = (j, k)$ be the unique node of $\lambda \setminus \nu$. Suppose that either $h \mid k$ or $(j+1, k-1)$ is not a node of λ. Define*

$$\begin{aligned} N = {} & 2 \times (\textit{number of addable nodes of } \lambda \textit{ of residue } \pm n \textit{ to the left of } \mathfrak{x} \textit{ and not in column } 1) \\ & + (\textit{number of addable nodes of } \lambda \textit{ to the left of } \mathfrak{x} \textit{ in column } 1) \\ & - 2 \times (\textit{number of removable nodes of } \lambda \textit{ of residue } \pm n \textit{ to the left of } \mathfrak{x}) \end{aligned}$$

and

$$\delta = \begin{cases} q + q^{-1} & (k \equiv 1 \ (\mathrm{mod}\ h),\ k > 1), \\ 1 & (\textit{otherwise}). \end{cases}$$

Then the coefficient of $|\lambda\rangle$ in $f_n|\nu\rangle$ is $q^N \delta$.

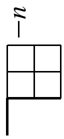
Proof. This also follows easily from the definitions in [12]. The given condition on $\mathfrak{x}$ means that we can only obtain λ from ν by adding the node $\mathfrak{x}$; we do not need to worry about adding a lower node and 'straightening' using the commutation rules. □

Now we want to examine the case where $m > 1$. For simplicity, we suppose that $\nu = \lambda^-$ and $\nu \xrightarrow{\pm n:m} \lambda$. In order to describe the coefficient of $|\lambda\rangle$ in $f_n^{(m)}|\nu\rangle$, we examine the possible configurations of columns of residue $\pm n$ and the columns adjacent to these in λ and ν. In the diagrams that follow, we draw portions of the Young diagram of λ, in which we use the following conventions:

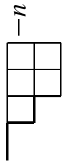
- columns of residue $\pm n$ are labelled with their residues;
- nodes of λ which are not nodes of ν are denoted with $\boxtimes$;
- the edges of the Young diagrams of λ and ν are marked with a bold line;
- the symbol indicates a (possibly empty) vertical stack of nodes.

For column 1, there are three possible configurations; we say that the 0-configuration of λ is A, B or C, according to which of the following occurs.

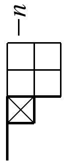
$A \quad \lambda'_1 = \nu'_1 = \nu'_2$



$B \quad \lambda'_1 = \nu'_1 = \nu'_2 + 1$

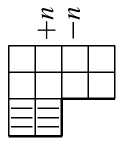


$C \quad \lambda'_1 = \nu'_1 + 1$

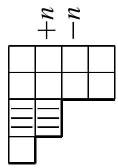


For $j \geqslant 1$, there are seven possible configurations for the pair of columns jh, $jh+1$. We say that the j-configuration of λ is D, E, F, G, H, I or J, where these configurations are as follows.

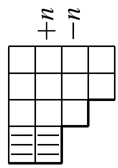
D $\quad \lambda'_{hj-1} = \lambda'_{hj} = \nu'_{hj}, \quad \lambda'_{hj+1} = \nu'_{hj+1} = \nu'_{hj+2}$



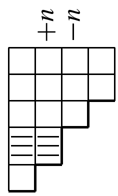
E $\quad \lambda'_{hj-1} = \lambda'_{hj} + 1 = \nu'_{hj} + 1, \quad \lambda'_{hj+1} = \nu'_{hj+1} = \nu'_{hj+2}$



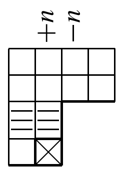
F $\quad \lambda'_{hj-1} = \lambda'_{hj} = \nu'_{hj}, \quad \lambda'_{hj+1} = \nu'_{hj+1} = \nu'_{hj+2} + 1$



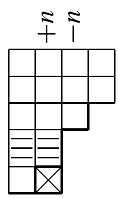
G $\quad \lambda'_{hj-1} = \lambda'_{hj} + 1 = \nu'_{hj} + 1, \quad \lambda'_{hj+1} = \nu'_{hj+1} = \nu'_{hj+2} + 1$



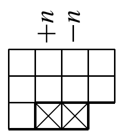
H $\quad \lambda'_{hj-1} = \lambda'_{hj} = \nu'_{hj} + 1, \quad \lambda'_{hj+1} = \nu'_{hj+1} = \nu'_{hj+2}$



I $\quad \lambda'_{hj-1} = \lambda'_{hj} = \nu'_{hj} + 1, \quad \lambda'_{hj+1} = \nu'_{hj+1} = \nu'_{hj+2} + 1$

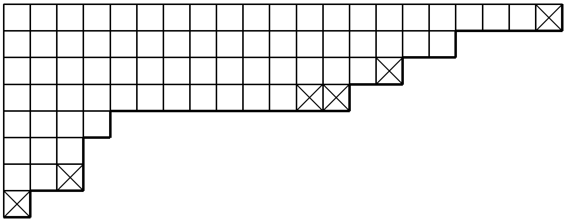


J $\quad \lambda'_{hj-1} = \lambda'_{hj} = \lambda'_{hj+1} = \nu'_{hj} + 1 = \nu'_{hj+1} + 1 = \nu'_{hj+2} + 1$



Now, if x and y represent any two of the letters A–J, we define $n_\lambda[x, y]$ to be the number of ordered pairs $a < b$ such that the a-configuration of λ is x and the b-configuration is y.

Example. Suppose $n = 1$, and let $\lambda = (21, 17, 15, 13, 4, 3^2, 1)$. The Young diagram of λ, with the nodes of $\lambda \setminus \lambda^-$ marked with $\boxtimes$, is as follows.



The configurations appearing from left to right are $C, I, D, D, J, H, E, H, D, D, D, \ldots$. So we have

$$\begin{gathered}
n_\lambda[C, I] = n_\lambda[C, J] = n_\lambda[C, E] = n_\lambda[I, J] = n_\lambda[I, E] = n_\lambda[J, E] \\
= n_\lambda[H, E] = n_\lambda[H, H] = n_\lambda[E, H] = 1, \\
n_\lambda[C, H] = n_\lambda[I, H] = n_\lambda[D, J] = n_\lambda[D, E] = n_\lambda[J, H] = 2, \\
n_\lambda[D, H] = 4, \\
n_\lambda[C, D] = n_\lambda[I, D] = n_\lambda[D, D] = n_\lambda[J, D] = n_\lambda[H, D] = n_\lambda[E, D] = \infty, \\
n_\lambda[x, y] = 0 \quad \text{for any other } x, y.
\end{gathered}$$

We extend the notation $n_\lambda[x, y]$ linearly in both variables, so for example $n_\lambda[2x_1 - x_2, y_1 - y_2]$ will mean $2n_\lambda[x_1, y_1] - n_\lambda[x_2, y_1] - 2n_\lambda[x_1, y_2] + n_\lambda[x_2, y_2]$.

Proposition 3.8. *Suppose $\lambda \in \mathrm{DP}_h$, $\nu = \lambda^-$ and $\nu \xrightarrow{\pm n:m} \lambda$. Define*

$$N(\nu, \lambda) = n_\lambda[A - B + 2E - 2F + H - I, H + I + 2J].$$

Then the coefficient of $|\lambda\rangle$ in $f_n^{(m)}|\nu\rangle$ is $q^{N(\nu,\lambda)}$.

Proof. We proceed by induction on m, with the case $m = 1$ being a special case of Lemma 3.7; the fact that the node removed lies in the outermost ladder of λ guarantees the hypotheses of that lemma.

Suppose $m > 1$, and let M be the set of partitions μ such that $\nu \xrightarrow{\pm n:m-1} \mu \xrightarrow{\pm n:1} \lambda$. For each $\mu \in M$ we have $\mu^- = \nu$, and so we may apply the inductive hypothesis to μ. The coefficient of $|\lambda\rangle$ in $f_n|\mu\rangle$ is $q^{N_\mu}\delta_\mu$, where N_μ and δ_μ are as in Lemma 3.7. So by induction we need to show that

$$\sum_{\mu \in M} q^{N(\nu,\mu)} q^{N_\mu} \delta_\mu = [m]_q q^{N(\nu,\lambda)}.$$

We take $\mu \in M$, and calculate $q^{N(\nu,\mu) - N(\nu,\lambda) + N_\mu}\delta_\mu$. If $\mathfrak{x} = (j, k)$ is the unique node of $\lambda \setminus \mu$, then the configuration in λ of the columns surrounding $\mathfrak{x}$ is either C, H, I or J, and the corresponding configuration in μ is A, E, G or H, respectively. If y is any of the letters A–J, then we write left($\mathfrak{x}$, y) for the number of configurations of type y to the left of $\mathfrak{x}$, and right($\mathfrak{x}$, y) similarly, and we extend these functions linearly in y. Then we have

$$N(\nu,\mu) - N(\nu,\lambda) = -\operatorname{left}(\mathfrak{x}, A - B + 2E - 2F + H - I) + \operatorname{right}(\mathfrak{x}, H + I + 2J).$$

The factor N_μ is easily seen to equal left($\mathfrak{x}$, $A - B - C + 2E - 2F - 2I - 2J$); adding this to the previous line yields

$$N(\nu,\mu) - N(\nu,\lambda) + N_\mu = \operatorname{right}(\mathfrak{x}, H + I + 2J) - \operatorname{left}(\mathfrak{x}, C + H + I + 2J).$$

Note that right($\mathfrak{x}$, $H + I + 2J$) is simply the number $r(\mathfrak{x})$ of nodes of $\lambda \setminus \nu$ to the right of $\mathfrak{x}$, while left($\mathfrak{x}$, $C + H + I + 2J$) is the number $l(\mathfrak{x})$ of nodes of $\lambda \setminus \nu$ to the left of $\mathfrak{x}$, or $l(\mathfrak{x}) - 1$ in the case where $k \equiv 1 \pmod h$ and $k > 1$ (that is, if the configuration of the columns surrounding $\mathfrak{x}$ is J; in this case, $l(\mathfrak{x})$ counts the node immediately to the left of $\mathfrak{x}$, but left($\mathfrak{x}$, $C + H + I + 2J$) does not). Hence we have

$$q^{N(\nu,\mu)-N(\nu,\lambda)} q^{N_\mu} \delta_\mu = \begin{cases} q^{r(\mathfrak{x})-l(\mathfrak{x})+1}(q + q^{-1}) & (k \equiv 1 \pmod h,\ k > 1), \\ q^{r(\mathfrak{x})-l(\mathfrak{x})} & \text{(otherwise)}. \end{cases}$$

Summing this over all $\mu \in M$ amounts to summing $q^{r(\mathfrak{x})-l(\mathfrak{x})}$ over all nodes of $\lambda \setminus \nu$ (not just those which are removable nodes of λ), and this gives (examining these nodes from left to right)

$$q^{m-0} + q^{(m-1)-1} + \cdots + q^{1-(m-1)} + q^{0-m} = [m]_q,$$

as required. □

Proof of Proposition 3.5 *(case $k = n$).* By Proposition 3.8, we need to show that

$$t(\lambda) - t(\lambda^-) = n_\lambda[A - B + 2E - 2F + H - I, H + I + 2J].$$

Let $X = \lambda \setminus \lambda^-$. First we consider semi-shallow ah-bars in λ and λ^-. Given j with $\lambda_j > 0$, let $\mathfrak{h}$ be the node at the end of row j of λ.

Claim 1. *If λ^- has a semi-shallow ah-bar in row j, then so does λ.*

Proof. Suppose λ^- has a semi-shallow ah-bar in row j. Then $\lambda^-_j \equiv 0 \pmod h$ and $\lambda^-_j > 0$. If $\lambda_j > \lambda^-_j$, then we must have $\lambda_j = \lambda^-_j + 1$, with the node $\mathfrak{h} = (j, \lambda_j) \in X$. But then the node $(j, \lambda_j - 1)$ also lies in X, which means that $\lambda^-_j = \lambda_j - 2$; contradiction. So $\lambda_j = \lambda^-_j$, and the node $\mathfrak{h} = (j, \lambda_j)$ does not lie in X.

We have $\lambda^-_{j+a} = \lambda_j - ah$; furthermore, if $\lambda_j > ah$ then the node $(j + a, \lambda_j - ah + 1)$ lies in the same ladder as $\mathfrak{h}$, and so cannot lie in X. Therefore, $\lambda_{j+a} = \lambda^-_{j+a} = \lambda_j - ah$, so λ has a semi-shallow ah-bar in row j. □

Claim 2.

(1) *If λ has a semi-shallow ah-bar in row j but λ^- does not, then $\mathfrak{h} \in X$ and $\lambda_j \equiv 0 \pmod h$.*

(2) *Suppose* $\lambda_j = bh$. *Then the statement*

$$\lambda \text{ has a semi-shallow } ah\text{-bar in row } j \text{ but } \lambda^- \text{ does not}$$

holds if and only if the $(b-a)$*-configuration of* λ *is* A, B, H *or* I, *while the* b*-configuration of* λ *is* H *or* I, *with the node marked* $\boxtimes$ *occurring in row* j.

Proof. (1) Certainly $\lambda_j \equiv 0 \pmod h$. If $\mathfrak{h} \notin X$, then we have $\lambda_j^- = \lambda_j$. If $\lambda_j > ah$, then the node $(j+a, \lambda_j - ah)$ lies in the same ladder as $\mathfrak{h}$, and so cannot lie in X. But this gives $\lambda_{j+a}^- = \lambda_{j+a} = \lambda_j^- - ah$, so that λ^- has a semi-shallow ah-bar in row j; contradiction.

(2) This is a matter of checking through the possible pairs of configurations. The fact that $\mathfrak{h} = (j, bh) \in X$ means that the b-configuration of λ must be H or I, with the $\boxtimes$ node being $\mathfrak{h}$. The fact that $\lambda_{j+a} = \lambda_j - ah$ means that either $b = a$ (in which case the $(b-a)$-configuration is A or B) or the node $(j+a, h(b-a))$ lies in X, in which case the $(b-a)$-configuration is H or I. □

Claim 3. *The number of semi-shallow bars of* λ *minus the number of semi-shallow bars of* λ^- *is*

$$n_\lambda[A+B+H+I, H+I].$$

Proof. The number required is the number of pairs j, a such that λ has a semi-shallow ah-bar in row j but λ^- does not. By Claim 2, such pairs correspond to pairs $(b-a, b)$ such that the $(b-a)$-configuration is A, B, H or I, while the b-configuration is H or I. □

Now we look at shallow bars in λ and λ^-.

Claim 4.

(1) *If* λ *has a shallow* ah*-bar in row* j *and* λ^- *does not, then we have* $\mathfrak{h} \in X$ *and* $\lambda_j \equiv 0$ *or* $1 \pmod h$.
(2) *Suppose* $\lambda_j = bh$ *or* $bh+1$. *Then the condition*

$$\lambda \text{ has a shallow } ah\text{-bar in row } j \text{ but } \lambda^- \text{ does not}$$

holds if and only if the $(b-a)$*-configuration and the* b*-configuration of* λ *form one of the ordered pairs* (A, J), (H, J), (E, H), (E, I), (E, J), (G, H) *or* (G, I), *with the nodes marked* $\boxtimes$ *in the* b*-configuration occurring in row* j.

Proof. 1. Suppose $\mathfrak{h} \notin X$. Then $\lambda_j^- = \lambda_j$, and $\lambda_{j+a}^- \leqslant \lambda_{j+a} < \lambda_j - ah$. So the only way λ^- can fail to have a shallow ah-bar in row j is if $\lambda_{j+b}^- = \lambda_j^- - ah$ for some $0 < b < a$. But if this happens, then the node $(j+b, \lambda_{j+b}^- + 1)$ lies in an earlier ladder than $\mathfrak{h}$, and so cannot possibly lie in X. So $\lambda_{j+b} = \lambda_{j+b}^- = \lambda_j - ah$; contradiction.

So we have $\mathfrak{h} \in X$, which gives $\lambda_j \equiv 0$ or $1 \pmod h$.

2. This is a matter of checking through all the possible pairs of configurations. □

In a similar way, we prove the following.

Claim 5.

(1) *If λ^- has a shallow ah-bar in row j and λ does not, then we have $\mathfrak{h} \in X$ and $\lambda_j \equiv 1 \pmod h$.*
(2) *Suppose $\lambda_j = bh + 1$. Then the condition*

$$\lambda^- \textit{ has a shallow } ah\textit{-bar in row } j \textit{ but } \lambda \textit{ does not}$$

holds if and only if the $(b-a)$-configuration of λ is F and the b-configuration of λ is J.

We deduce the following, which is proved in the same way as Claim 3.

Claim 6. *The number of shallow bars of λ minus the number of shallow bars of λ^- equals*

$$n_\lambda[A - F + H, J] + n_\lambda[E, H + I + J] + n_\lambda[G, H + I].$$

Finally we consider h-bars. Given $j < l$ with $\lambda_j > 0$, let $\mathfrak{h}$ be the node at the end of row j of λ. The following results are proved in the same way as the results above.

Claim 7.

(1) *If λ has a shallow h-pair in rows j and l but λ^- does not, then we have $\mathfrak{h} \in X$, $\lambda_j \equiv 1 \pmod h$ and $\lambda_l \equiv -1 \pmod h$.*
(2) *Suppose $\lambda_j = bh + 1$ and $\lambda_l = ah - 1$. Then the condition*

$$\lambda \textit{ has a shallow } h\textit{-pair in rows } j \textit{ and } l \textit{ but } \lambda^- \textit{ does not}$$

holds if and only if the a-configuration of λ is E and the b-configuration of λ is J.

Claim 8.

(1) *If λ^- has a shallow h-pair in rows j and l but λ does not, then we have $\mathfrak{h} \in X$, $\lambda_j \equiv 0$ or $1 \pmod h$ and $\lambda_l \equiv 1 \pmod h$.*
(2) *Suppose $\lambda_j = bh + 1$ and $\lambda_l = ah - 1$. Then the condition*

$$\lambda^- \textit{ has a shallow } h\textit{-pair in rows } j \textit{ and } l \textit{ but } \lambda \textit{ does not}$$

holds if and only if the a-configuration and the b-configuration of λ form one of the ordered pairs (B, H), (B, I), (B, J), (F, H), (F, I), (F, J), (G, H), (G, I), (I, H), (I, I) or (I, J), with the nodes marked $\boxtimes$ lying in row j.

Claim 9. *The number of shallow h-pairs in λ minus the number of shallow h-pairs in λ^- equals*

$$n_\lambda[E, J] - n_\lambda[B + F + I, H + I + J] - n_\lambda[G, H + I].$$

Combining Claims 3, 6 and 9, we find that

$$t(\lambda) - t(\lambda^-) = n_\lambda[A - B + 2E - 2F + H - I, H + I + 2J],$$

and Proposition 3.8 gives the result. □

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